الفضاءات التبولوجية الثلاثية
عند بارا – ليندوف و سيمبيرا – ليندوف

On Para-Lindelöf and Semipara-Lindelöf Tritopological Spaces

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Abstract:

We define a para-Lindelöf tritopological Space and Semipara-Lindelöf tritopological space and we find some properties of these concepts and give the relation between these concepts .

Introduction:

The concept of paracompactness is due to Dieudonne [4].The concept of para- Lindelöf is due to Fleissner [5].The term space (X,τ,ρ) is referred to as a set X with two general non identical topologies τ and ρ [2].The term space (X,τ,ρ,μ) is referred to as as set X with three general non identical topologies τ , ρ and μ[3] .Kovar [3] introduce the concept of paracompactness in the tritopological (3topological) space.Al-Fatlawee [1] study the concept of paracompactness in bitopological spaces and tritopological space and give several results and properties of this concept.A collection of subsets of X is locally finite (resp.locally countable) [5] with respect to the topology ρ if every $x \in X$ has a ρ-neighborhood meeting finitely many (resp.countable many) elements of the collection.A collection has the σ-property [5] if it is the union of countably

many collection with the property .A cover (or covering)of a space (X,τ) [4] is a collection of subset of X whose union is all of X .A τ-open cover of X is a cover consisting of τ-open sets ,and other adjective applying to subsets apply similarly to cover .If $\prod$ and Φ are covers of X ,we say Φ refines $\prod$ [4] if each number of Φ is contained in some member of $\prod$ .Then we say Φ refines (or refinement of) $\prod$ .A subset of a topological space (X,τ) is an F_σ with respect to the topology [4] if it is a countable union of τ-closed sets ,and written by τ-F_σ .

2- Para-Lindelöf Tritopological Spaces

2.1. Definition [1]

A bitopological space (X,τ,ρ) is said to be $(\tau-\rho)$-compact (resp. Lindelöf) with respect to ρ if every τ-open cover has a finite (resp.countable) ρ-open subcover.

2.2. Definition [3]

A tritopological space (X,τ,ρ,μ) is said to be $(\tau-\rho)$-paracompact with respect to μ if every τ-open cover has a ρ-open refinement which is locally finite with respect to μ.

2.3.Definition

A tritopological space (X,τ,ρ,μ) is said to be $(\tau-\rho)$-para-Lindelöf with respect to μ if every τ-open cover has a ρ-open refinement which is locally countable with respect to μ.

2.4.Proposition:

If a tritopological space (X,τ,ρ,μ) is $(\tau-\rho)$-Lindelöf with respect to μ then it is $(\tau-\rho)$-para-Lindelöf with respect to μ.

Proof:

This follows from the fact that every countable collection is locally countable .

2.5.Proposition:

If a tritopological space (X,τ,ρ,μ) is $(\tau-\rho)$-paracompact with respect to μ then it is $(\tau-\rho)$-para-Lindelöf with respect to μ

Proof

This follows from the fact that every locally finite collection is locally countable .

2.6.Theorem :

If (X,τ,ρ,μ) is $(\tau-\rho)$-para-Lindelöf with respect to μ then the τ-closed subspace (Y,τ_Y,ρ_Y,μ_Y) is $(\tau_Y-\rho_Y)$-para-Lindelöf with respect to μ_Y .

Proof

Let $\Phi=\{U_\lambda:\lambda\in\Lambda\}$ be a τ_Y-open cover of Y .Since each U_λ is a τ_Y-open subset of Y ,there is τ-open subset V_λ of X such that $U_\lambda=V_\lambda\cap Y$ for each $\lambda\in\Lambda$.Let $\prod=\{V_\lambda:\lambda\in\Lambda\}\cup\{X/Y\}$ .Then $\prod$ is τ-open cover of X .By hypothesis $\prod$ has a ρ-open refinement $\Psi=\{W_\gamma:\gamma\in\Gamma\}$ which is locally countable with respect to μ .Set $\Omega=\{W_\gamma\cap Y:\gamma\in\Gamma\}$.Then Ω is ρ_Y-open refinement of Φ which is locally countable with respect to μ_Y .

2.7.Theorem:

Let (X,τ,ρ,μ) be a tritopological space ,and let $\Sigma=\{X_i:X_i\in\tau\cap\rho\cap\mu\}$be a partition of X .The space (X,τ,ρ,μ) is $(\tau-\rho)$-para-Lindelöf with respect to μ if and only if the space (X,τ_i,ρ_i,μ_i) is $(\tau_i-\rho_i)$-para-Lindelöf with respect to μ_i .

Proof

The "only if part ",since $X_i = X / \bigcup_{i \neq j} X_j$ is τ-closed then the subspace $(X, \tau_i, \rho_i, \mu_i)$ is $(\tau_i - \rho_i)$-para-Lindelöf with respect to μ_i for every i by Theorem 2.6.

The " if part ".Let $\Phi = \{U_\lambda : \lambda \in \Lambda\}$ be a τ-open cover of X .The collection $\prod = \{U_\lambda \cap X_i : \lambda \in \Lambda\}$ be a τ_i-open cover of X_i with cardinality $\leq m$ every i.By hypothesis $\prod$ has a ρ_i-open refinement $\Psi_i = \{W_{i\lambda} : \lambda \in \Lambda\}$ which is locally countable with respect to μ_i .Let $\Omega = \left\{ \bigcup_{i \in I} W_{i\lambda} : \lambda \in \Lambda \right\}$ .Then Ω is ρ-open refinement of Φ which is locally countable with respect to μ .

2.8.Theorem:

If each τ-open set in a $(\tau - \rho)$-para-Lindelöf with respect to μ space (X, τ, ρ, μ) is $(\tau - \rho)$-para-Lindelöf with respect to μ ,then every subspace $(Y, \tau_Y, \rho_Y, \mu_Y)$ is $(\tau_Y - \rho_Y)$-para-Lindelöf with respect to μ_Y .

Proof

Let $\Phi = \{U_\lambda : \lambda \in \Lambda\}$ be a τ_Y-open cover of Y .Since each U_λ is a τ_Y-open subset of Y ,there is a τ-open subset V_λ of X such that $U_\lambda = V_\lambda \cap Y$ for each $\lambda \in \Lambda$ .Then $G = \bigcup_{\lambda \in \Lambda} V_\lambda$ is a τ-open set.

Let $\Omega = \{V_\lambda : \lambda \in \Lambda\}$ be a τ-open cover of G .By hypothesis G is $(\tau - \rho)$-para-Lindelöf with respect to μ ,thus Ω has a ρ-open refinement $\Psi = \{W_\gamma : \gamma \in \Gamma\}$ which is locally countable with respect to μ .Set $\Sigma = \{W_\gamma \cap Y : \gamma \in \Gamma\}$ .Then Σ is ρ_Y-open refinement of Φ which is locally countable with respect to μ_Y .

3. Semipara-Lindelöf Tritopological Spaces

3.1. Definition [3]

A tritopological space (X,τ,ρ,μ) is said to be $(\tau-\rho)$-semiparacompact with respect to μ ,if each τ-open cover of X has a τ-open refinement which is σ-locally finite with respect to μ.

3.2. Definition

A tritopological space (X,τ,ρ,μ) is said to be $(\tau-\rho)$-semipara-Lindelöf with respect to μ,if each τ-open cover of X has a ρ-open refinement which is σ-locally countable with respect to μ.

3.3. Proposition

If a tritopological space (X,τ,ρ,μ) is $(\tau-\rho)$-para-Lindelöf with respect to μ then it is $(\tau-\rho)$-semipara-Lindelöf with respect to μ.

Proof

This follows from the fact that every locally countable collection is σ-locally countable.

3.4. Proposition [1]

If a tritopological space (X,τ,ρ,μ) is $(\tau-\rho)$-paracompact with respect to μ then it is $(\tau-\rho)$-semiparacompact with respect to μ.

3.5. Proposition

If a tritopological space (X,τ,ρ,μ) is $(\tau-\rho)$-semiparacompact with respect to μ then it is $(\tau-\rho)$-semipara-Lindelöf with respect to μ.

Proof

This follows from the fact that every σ-locally finite collection is σ-locally countable.

3.6. Corollary

If a tritopological space (X,τ,ρ,μ) is $(\tau-\rho)$-paracompact with respect to μ then it is $(\tau-\rho)$-semipara-Lindelöf with respect to μ.

Proof

This follows from Proposition 3.4.and Proposition 3.5.

3.7.Theorem

If a tritopological space (X,τ,ρ,μ) is $(\tau-\rho)$- semipara-Lindelöf with respect to μ then the τ-closed subspace (Y,τ_Y,ρ_Y,μ_Y) is $(\tau_Y-\rho_Y)$-semipara-Lindelöf with respect to μ_Y.

Proof

Let $\Phi=\{U_\lambda:\lambda\in\Lambda\}$ be a τ_Y-open cover of Y.Since each U_λ is a τ_Y-open subset of Y,there is a τ-open subset V_λ of X such that $U_\lambda=V_\lambda\cap Y$.Let $\prod=\{V_\lambda:\lambda\in\Lambda\}\cup\{X/Y\}$.Then $\prod$ is τ-open cover of X.By hypothesis $\prod$ is a ρ-open refinement Ω which is σ-locally countable with respect to μ ,hence $\Omega=\bigcup_n\Omega_n$ where each $\Omega_n=\{W_{n\gamma}:\gamma\in\Gamma\}$ is locally countable with respect to μ.Set $\Psi=\bigcup_n\Psi_n$ where each $\Psi_n=\{W_{n\gamma}\cap Y:\gamma\in\Gamma\}$.Then Ψ is ρ-open refinement of Φ which is σ-locally countable with respect to μ.

3.8.Theorem

Let (X,τ,ρ,μ) be a tritopological space ,and let $\Sigma=\{X_i:X_i\in\tau\cap\rho\cap\mu\}$ be a partition of X .The space (X,τ,ρ,μ)

is $(\tau-\rho)$-semipara-Lindelöf with respect to μ if and only if the space (X,τ_i,ρ_i,μ_i) is $(\tau_i-\rho_i)$-para-Lindelöf with respect to μ_i.

Proof

The "only if part ",since $X_i = X / \bigcup_{i\neq j} X_j$ is τ-closed then the subspace (X,τ_i,ρ_i,μ_i) is $(\tau_i-\rho_i)$-semipara-Lindelöf with respect to μ for every i by Theorem 3.7.

The " if part ".Let $\Phi=\{U_\lambda : \lambda\in\Lambda\}$ be a τ-open cover of X .The collection $\prod=\{U_\lambda \cap X_i : \lambda\in\Lambda\}$ be a τ_i-open cover of X_i for every i. By hypothesis $\prod$ has a ρ_i-open refinement $\Psi_i=\{W_{i\lambda} : \lambda\in\Lambda\}$ which is $\sigma-$locally countable with respect to μ_i .So $\Psi_i=\bigcup_n \Psi_{in}$ where each $\Psi_{in}=\{W_{in\lambda} : \lambda\in\Lambda\}$ is locally countable with respect to μ .Set $\Omega=\bigcup_n \Omega_n$ where .Then Ω is ρ-open refinement of Φ which is σ-locally countable with respect to μ.

3.9.Theorem

If (X,τ,ρ,μ) is $(\tau-\rho)$-semipara-Lindelöf with respect to μ, then the $\tau-F_\sigma$ subspace (Y,τ_Y,ρ_Y,μ_Y) is $(\tau_Y-\rho_Y)$-semipara-Lindelöf with respect to μ_Y.

Proof

Let $\Phi=\{U_\lambda : \lambda\in\Lambda\}$be a τ_Y-open cover of Y .Since each U_λ is a τ_Y-open cover of Y ,there exists a τ-open subset V_λ of X such that $U_\lambda = V_\lambda \cap Y$ for each fixed n the collection $\prod_n=\{V_\lambda : \lambda\in\Lambda\}\cup\{X / Y_n\}$form a τ-open cover of X .By hypothesis $\prod_n$ has a ρ-open refinement Ω which is σ-locally countable with respect to μ . Then $\Omega=\bigcup_n \Omega_n$ where each

$\Omega_n = \{W_{n\gamma} : \gamma \in \Gamma\}$is locally countable with respect to μ. For each n, let $\Psi = \bigcup_n \Psi_n$ such that $\Psi_n = \{W_{n\gamma} \cap Y : W_{n\gamma} \cap Y \neq \phi, \gamma \in \Gamma\}$.Then Ψ is ρ_Y-open refinement of Φ which is σ-locally countable with respect to μ_Y.

3.10.Theorem

If (X, τ, ρ, μ) is $(\tau - \rho)$-para-Lindelöf with respect to μ, then the $\tau - F_\sigma$ subspace $(Y, \tau_Y, \rho_Y, \mu_Y)$ is $(\tau_Y - \rho_Y)$-semipara-Lindelöf with respect to ρ_Y.

Proof

Suppose that Y is $\tau - F_\sigma$ set .Then $Y = \bigcup_n Y_n$,where each Y_n is τ-closed .Let $\Phi = \{U_\lambda : \lambda \in \Lambda\}$be a τ_Y-open cover of Y .Since each U_λ is a τ_Y-open cover of Y,there exists a τ-open subset V_λ of X such that $U_\lambda = V_\lambda \cap Y$ for each λ. For each fixed n the collection $\prod_n = \{V_\lambda : \lambda \in \Lambda\} \cup \{X / Y_n\}$form a τ-open cover of X .By hypothesis $\prod_n$ has a ρ-open refinement Ω which is σ-locally countable with respect to μ . Then $\Omega = \{W_{\lambda n} : (\lambda, n) \in \Lambda \times N\}$is locally countable with respect to ρ. For each n, let $\Psi_n = \{W_{n\gamma} \cap Y : W_{n\gamma} \cap Y \neq \phi, \gamma \in \Gamma\}$.Let $\Psi = \bigcup_n \Psi_n$ .Then Ψ is ρ_Y-open refinement of Φ which is σ-locally countable with respect to μ_Y.

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